Mathematical Finance
Dylan Possamaï

Assignment 5

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

Sub-martingales are good integrators

Show that $(\mathbb{F}, \mathbb{P})$–local sub-martingale is an $(\mathbb{F}, \mathbb{P})$–good integrator.

Hint: it will be useful here to use the Doob–Meyer decomposition for discrete-time martingales.

Khintchine inequalities

The goal of this exercise is to prove that if $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that the set

$$S := \bigg\{ \sum_{k=0}^{n} \varepsilon_k Z_k : n \in \mathbb{N},\ (\varepsilon_k)_{k\in\{0,\dots,n\}} \in \{-1,1\}^{n+1} \bigg\},$$

is bounded in $\mathbb{P}$-probability, then, $\sum_{k=0}^{\infty} Z_k^2$ is $\mathbb{P}$–a.s. finite. In particular, $Z_k \longrightarrow_{k\to+\infty} 0$, both $\mathbb{P}$–a.s. and in $\mathbb{P}$-probability.

1) In order to prove this result, we need first an intermediary lemma, which is a variant of the so–called Khintchine inequalities. Fix some $n \in \mathbb{N}$ and some sequence $(\varepsilon_k)_{k\in\{0,\dots,n\}}$ of $\mathbb{P}$-independent random variables which are uniformly distributed on $\{-1, 1\}$ under $\mathbb{P}$. We will prove first that for any $K \in (0, 1)$, there exists $\delta > 0$ such that for any $(\lambda_k)_{k\in\{0,\dots,n\}}$ taking values in $\mathbb{R}$

$$\mathbb{P}\Bigg[\bigg| \sum_{k=0}^{n} \lambda_k \varepsilon_k \bigg| \geq K \bigg(\sum_{k=0}^{n} \lambda_k^2 \bigg)^{1/2} \Bigg] \geq \delta.$$

   a) Show that for any random variable $Y \in \mathbb{L}^1(\mathbb{R}, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}^{\mathbb{P}}[Y] > 0$

$$\mathbb{P}[Y > 0] \geq \mathbb{E}^{\mathbb{P}}[Y]^2/\mathbb{E}^{\mathbb{P}}[Y^2].$$

   b) Verify that

$$\mathbb{E}^{\mathbb{P}}\Bigg[\bigg(\sum_{k=0}^{n} \lambda_k \varepsilon_k \bigg)^2 \Bigg] = \mathbb{V}\mathrm{ar}^{\mathbb{P}}\Bigg[\sum_{k=0}^{n} \lambda_k \varepsilon_k \Bigg] = \sum_{k=0}^{n} \lambda_k^2,$$

   and

$$\mathbb{E}^{\mathbb{P}}\Bigg[\bigg(\sum_{k=0}^{n} \lambda_k \varepsilon_k \bigg)^4 \Bigg] = \sum_{k=0}^{n+1} \lambda_k^4 + 3 \sum_{\{(j,k)\in\{0,\dots,n\}^2 : j\neq k\}} \lambda_k^2 \lambda_j^2 = 3 \bigg(\sum_{k=0}^{n} \lambda_k^2 \bigg)^2 - 2 \sum_{k=0}^{n+1} \lambda_k^4 \leq 3 \bigg(\sum_{k=0}^{n} \lambda_k^2 \bigg)^2.$$

   c) Using the two previous questions, deduce that whenever the $(\lambda_k)_{k\in\{0,\dots,n\}}$ are not all equal to 0, the desired result holds with $\delta = \frac{(1-K^2)^2}{3-2K^2+K^4}$. Conclude.

2) Prove that if μ is the uniform probability measure on $E := \{-1, 1\}^{n+1}$, then for any $K \in (0, 1)$, there is some $\delta > 0$ such that

$$\int_E \mathbf{1}_{\{|\Sigma_{k=0}^{n} \varepsilon_k Z_k| \geq K\sigma_n\}} \mathrm{d}\mu(\varepsilon) \geq \delta,$$

   where for any $n \in \mathbb{N}$, $\sigma_n := \big(\sum_{k=0}^{n} Z_k^2 \big)^{1/2}$.

3) Deduce that for any $L > 0$

$$\int_E \mathbb{P}\left[\left|\sum_{k=0}^{n} \epsilon_k Z_k\right| \geq K\sigma_n \geq KL\right] \mathrm{d}\mu(\varepsilon) \geq \delta\mathbb{P}[\sigma_n \geq L],$$

3) Show that for any $\beta > 0$, we have for any $L > 0$ large enough

$$\beta \geq \delta\mathbb{P}[\sigma_n \geq L].$$

Conclude.